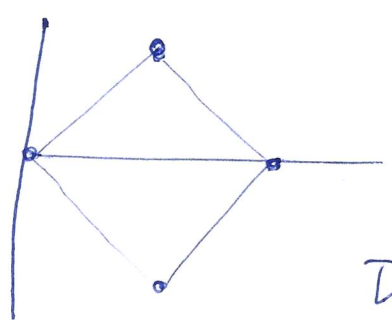
- Use an appropriate change of variable to evaluate

$$\iint_D (x-y)^2\,dx\,dy \;,$$

MATH 20E
FINAL REVIEW

where D is the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$.



We find that the equations of the four lines defining D are

$x-y=0$, $x-y=2$, $x+y=0$, $x+y=2$.

This suggests to write $\begin{cases} u = x-y \\ v = x+y \end{cases}$.

In other words, $x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2}$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

The region D^* in the (u,v)-plane is defined by $\begin{cases} 0 \le u \le 2 \\ 0 \le v \le 2 \end{cases}$, hence,

$$\iint_D (x-y)^2\,dx\,dy = \frac{1}{2}\iint_{D^*} u^2\,du\,dv = \frac{1}{2}\int_0^2\int_0^2 u^2\,du\,dv$$

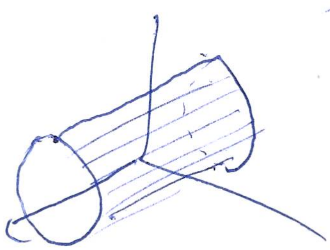
$$= \frac{1}{2}\times 2\times\left[\frac{u^3}{3}\right]_0^2 = \boxed{\frac{8}{3}}$$

- Let $W = \left\{ (x,y,z) \in \mathbb{R}^3 \mid y^2+z^2 \leq 1 \text{ and } -1 \leq x \leq 1 \right\}$,

and $\vec{F}(x,y,z) = \begin{pmatrix} y^2 z^2 \\ z^2 y \\ z y^2 \end{pmatrix}$. Find $\displaystyle\iint_{\partial W} \vec{F} \cdot d\vec{S}$

with ∂W oriented with outward-pointing normal

From Gauss, $\displaystyle\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F} \, dV$



$$= \iiint_W z^2 + y^2 \, dx\,dy\,dz$$

We use polar coordinates around the x-axis: $(x, r\cos\theta, r\sin\theta)$

$$\begin{cases} -1 \leq x \leq 1 \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\iiint_W z^2 + y^2 \, dx\,dy\,dz = \int_0^{2\pi} \int_{-1}^{1} \int_0^1 r^2 r \, dr \, dx \, d\theta$$

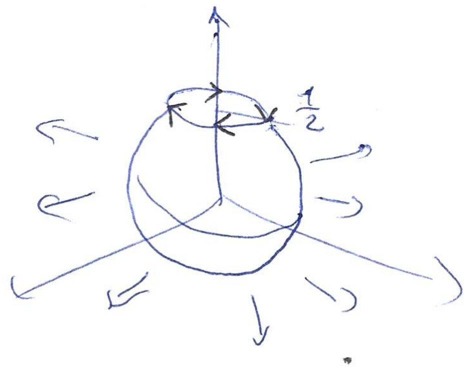
$$= 2 \times 2\pi \times \int_0^1 r^3 \, dr$$

$$= 4\pi \left[\frac{r^4}{4} \right]_0^1 = \boxed{\pi}$$

- let S be the portion of unit sphere defined by

$$\begin{cases} x^2+y^2+z^2=1 \\ z \leq \frac{1}{2} \end{cases}, \quad \text{with outward pointing normal } \vec{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

and $\vec{F}(x,y,z) = (-y, x, z)$. Find $\displaystyle\iint_S \nabla \times \vec{F} \cdot d\vec{S}$



From Stokes,

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

Oriented like on the picture!

∂S is defined by $\begin{cases} z = \frac{1}{2} \\ x^2+y^2 = \frac{3}{4} \end{cases}$, so a parametrization of it with the right orientation is

$$\theta \mapsto \left(\sqrt{\tfrac{3}{4}}\cos(2\pi-\theta), \sqrt{\tfrac{3}{4}}\sin(2\pi-\theta), \tfrac{1}{2}\right)$$

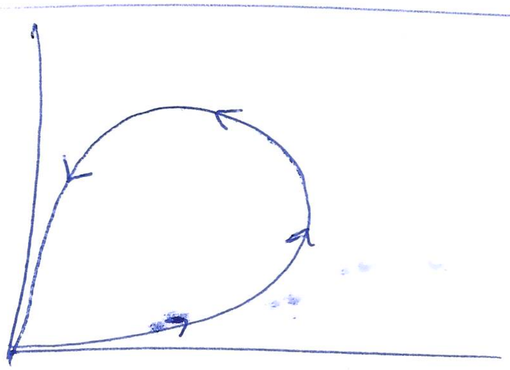
Instead, let's use $\vec{c}(\theta) = \left(\sqrt{\frac{3}{4}}\cos(\theta), \sqrt{\frac{3}{4}}\sin(\theta), \frac{1}{2}\right)$ and write

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = -\int_0^{2\pi} \vec{F}(\vec{c}(\theta)) \cdot c'(\theta)\, d\theta = -2\pi \times \frac{3}{4} = \boxed{-\frac{3\pi}{2}}$$

• Let C be the curve (Folium of Descartes) defined by

$$\vec{c}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right), \quad 0 \le t \le \infty$$

(Equation $x^3+y^3=3xy$). Find the area it encloses.



Choice of the version of Green you prefer:

$$\text{Area} \begin{cases} = \dfrac{1}{2}\displaystyle\int_{\vec{c}} x\,dy - y\,dx \\ = \displaystyle\int_{\vec{c}} x\,dy \\ = -\displaystyle\int_{\vec{c}} y\,dx \end{cases}$$

Picking the first one,

$$\text{Area} = \frac{1}{2}\int_0^\infty \frac{-3t^2 \times \left(3(1+t^3) - 3t \times 3t^2\right) + 3t \times \left(3 \times 2t \times (1+t^3) - 3t^2 \times 3t^2\right)}{(1+t^3) \times (1+t^3)^2}\,dt$$

$$= \frac{9}{2}\int_0^\infty \frac{(2t^2 + 2t^5 - 3t^5) - (t^2 + t^5 - 3t^5)}{(1+t^3)^3}\,dt = \frac{9}{2}\int_0^\infty \frac{t^2(1+t^3)}{(1+t^3)^3}\,dt$$

$$= \frac{9}{2}\int_0^\infty \frac{t^2}{(1+t^3)^2}\,dt = \frac{9}{2}\left[\frac{-1}{3(1+t^3)}\right]_0^\infty = \frac{9}{2} \times \frac{1}{3} = \boxed{\frac{3}{2}}$$